

VISIBILITY OF QUANTUM GRAPH SPECTRUM FROM THE VERTICES

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ABSTRACT. We investigate the relation between the eigenvalues of the Kirchhoff Laplacian on a finite metric graph and a corresponding Titchmarsh–Weyl function. We establish an explicit, optimal bound in terms of the edge lengths and the connectivity of the graph such that all eigenvalues below that bound are visible for the Titchmarsh–Weyl function. This bound corresponds to the smallest real resonance of the graph.

1. INTRODUCTION

Inverse problems for quantum graphs, i.e. differential operators on metric graphs, is an area which has attracted much interest during the last decade. Particular attention was paid to the problem of recovering a differential operator from the Titchmarsh–Weyl function on a selected set of vertices, see [1, 2, 7, 10, 13, 18, 19, 20, 22, 26, 27]. In contrast to the case of ordinary or partial differential operators the Titchmarsh–Weyl function for a quantum graph does not contain the full information on the differential operator on the graph. This effect is caused by the existence of real resonances for the graph obtained from attaching infinite leads to the vertices of G ; they materialize through eigenfunctions which vanish at all vertices of the graph. Their existence can be also be understood as a lack of a unique continuation property for quantum graphs.

In this paper we consider as a model operator the Laplacian $-\Delta_G$ on a finite metric graph G subject to continuity and Kirchhoff conditions on all vertices; see Section 2 for the details. Our aim is to study the relation between the (purely discrete) spectrum $\sigma(-\Delta_G)$ of $-\Delta_G$ and the poles of the Titchmarsh–Weyl function for a sufficiently large set of vertices. For any subset $B = \{v_1, \dots, v_m\}$ of the vertex set of G the matrix-valued Titchmarsh–Weyl function can be defined via the relation

$$M_B(\mu) \begin{pmatrix} \partial_\nu f(v_1) \\ \vdots \\ \partial_\nu f(v_m) \end{pmatrix} = \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_m) \end{pmatrix}, \quad \mu \in \mathbb{C} \setminus \sigma(-\Delta_G),$$

where f is any solution of $-f'' = \mu f$ on G which is continuous on each vertex and satisfies the Kirchhoff condition on any vertex which does not belong to B ; here $\partial_\nu f(v)$ is the sum of the derivatives at a vertex v of the restrictions of f to the edges incident to v . There is a close relation between the eigenvalues of $-\Delta_G$ and the poles of the meromorphic matrix function $\mu \mapsto M_B(\mu)$, see, e.g., [4, Section 3.5.3]. In fact, each pole of M_B is an eigenvalue of $-\Delta_G$, but the converse is not true in general if cycles with commensurate edges are present, see, e.g., [22].

In this paper we focus on the case where the vertex set B is rather large. We assume that B contains all vertices of degree one as well as all *proper core vertices*; cf. Definition 4.3. We define the number

$$\lambda_G = \inf \left\{ \frac{\nu(C)^2 \pi^2}{u_C^2} : C \text{ cycle in } G \text{ with commensurate edges} \right\}, \quad (1.1)$$

where $\nu(C)$ equals either 1 or 2 depending on properties of the edge lengths in C and u_C is the largest unit of length with respect to which the edges in C have integer lengths; see Section 3 for more detail. In the main result of this paper we show that any eigenvalue of $-\Delta_G$ below λ_G is visible (including its multiplicity) for the Titchmarsh–Weyl function: for any $\lambda < \lambda_G$ the equivalence

$$\lambda \in \sigma(-\Delta_G) \quad \text{if and only if} \quad \lambda \text{ is a pole of } M_B \quad (1.2)$$

holds and

$$\dim \ker(-\Delta_G - \lambda) = \text{rank Res}_\lambda M_B, \quad (1.3)$$

where $\text{Res}_\lambda M_B$ is the residue of the meromorphic matrix function M_B at λ ; cf. Section 4. Moreover, we show that this bound is optimal. In fact, we show that the number $\lambda = \lambda_G$ is an eigenvalue of $-\Delta_G$ such that (1.3) is violated whenever λ_G is finite, and we provide an example where λ_G is not a pole of M_B but a removable singularity. However, we would like to mention that eigenvalues satisfying (1.2) and (1.3) may also exist above λ_G .

In the proof of our main result Theorem 4.5 we trace the relations (1.2) and (1.3) back to considerations on resonances of G . The core piece of our proof is Theorem 3.1, which states that the number λ_G in (1.1) is in fact the smallest real resonance of G if λ_G is finite and that no real resonances exist if λ_G is infinite. We believe that Theorem 3.1 may be of independent interest.

We would like to mention that a connection between the existence of resonance eigenfunctions (so-called scars) and cycles with commensurate edges was already observed in [23]. For further work related to resonances and scars for quantum graphs we refer the reader to [3, 8, 11, 12, 15, 16, 24, 25] and the references therein. Moreover, we refer the reader to [5, 14] for genericity of simple eigenvalues and corresponding properties of edge lengths.

The present paper is organized as follows. In Section 2 we provide preliminaries on discrete and metric graphs, Laplacians and resonances. Section 3 contains Theorem 3.1 on the smallest real resonance as well as several examples which illustrate the calculation of λ_G and indicate that a large number of eigenvalues may lie below λ_G . Finally, in Section 4 we discuss the relation to the Titchmarsh–Weyl function and provide our main result Theorem 4.5.

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2. PRELIMINARIES

In this section we fix some notation and provide preliminaries on metric graphs and graph Laplacians.

2.1. Graphs, walks, and cycles. We use standard notation from graph theory, see, e.g., the standard textbook [6]. A (*discrete*) *graph* G consists of a nonempty set of edges E , a nonempty set of vertices V , and an incidence function ψ which assigns an unordered pair of (not necessarily distinct) vertices to each edge; if a vertex $v \in V$ is a member of the pair $\psi(e)$ for an edge $e \in E$ we say that e is *incident* to v . Furthermore, we say that an edge e is a *loop* if the two members of $\psi(e)$ coincide. The *degree* of a vertex $v \in V$ is the number of edges incident to v where each loop is counted twice. A *walk* in G is a tuple

$$W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{w-1}, e_w, v_w) \quad (2.1)$$

whose entries are alternately vertices and edges such that for each $k \in \{1, \dots, w\}$ the vertices v_{k-1} and v_k are precisely those two to which e_k is incident. For two walks W_1, W_2 in G we write $W_1 \subset W_2$ if W_1 is a subtuple consisting of consecutive entries of W_2 . A walk W of the form (2.1) is called *closed* if $v_0 = v_w$ holds. If, in addition, the vertices v_1, \dots, v_w are distinct then W is called a *cycle*. A graph G is called *connected* if for each two vertices v, \hat{v} there exists a walk in G starting at v and terminating at \hat{v} ; for simplicity throughout this paper we will only consider connected graphs. Furthermore, G is called *finite* if the sets V and E are finite. In this paper we consider only finite graphs and denote by $|V|$ and $|E|$ the number of vertices and edges, respectively, of G .

2.2. Metric graphs, Kirchhoff Laplacians, and resonances. Let us shortly recall some fundamentals on Laplacians on metric graphs; for more details see the recent monographs [4, 21] or the survey [17]. A *metric graph* is a graph G together with a length function $L : E \rightarrow (0, \infty)$ such that each edge $e \in E$ is identified with the interval $[0, L(e)]$, where the endpoints 0 and $L(e)$ are identified with the two vertices to which e is incident. In the following all graphs are metric graphs and we just write $e = [0, L(e)]$ for any edge e . For an edge $e \in E$ and a vertex $v \in V$ we write $v = o(e)$ or $v = t(e)$ if the left or right endpoint of e , respectively, is identified with the vertex v and say that e originates from or terminates at v , respectively. Note that if e is a loop then $o(e) = t(e)$ holds.

Let G be a finite, connected metric graph. On G we consider the Hilbert space $L^2(G)$ of (equivalence classes of) square-integrable functions $f : G \rightarrow \mathbb{C}$, equipped with the standard norm and inner product. For any $f \in L^2(G)$ and any edge $e \in E$ we denote by f_e the restriction of f to e , and we identify f_e with a function on $[0, L(e)]$. Moreover, we write

$$\tilde{H}^k(G) = \{f \in L^2(G) : f_e \in H^k(e), e \in E\}, \quad k = 1, 2, \dots,$$

where $H^k(e)$ is the Sobolev space of order k on $(0, L(e))$. Note that any $f \in \tilde{H}^k(G)$ is continuous on $(0, L(e))$ and $f|_{(0, L(e))}$ has a continuous extension to $[0, L(e)]$ for each $e \in E$. We say that a function $f \in \tilde{H}^1(G)$ is *continuous on G* if for each two edges e and \hat{e} which are incident to a joint vertex v the limits of f_e and $f_{\hat{e}}$ at v coincide. Accordingly we define

$$H^1(G) = \{f \in \tilde{H}^1(G) : f \text{ is continuous on } G\}.$$

For $f \in H^1(G)$ we write $f(v)$ for the limit of f_e at v for an arbitrary edge e incident to v . Moreover, for $f \in H^1(G) \cap \tilde{H}^2(G)$ we use the abbreviation

$$\partial_\nu f(v) = \sum_{t(e)=v} f'_e(L(e)) - \sum_{o(e)=v} f'_e(0), \quad v \in V,$$

for the sum of the derivatives pointing outwards; here the evaluations of the derivatives must be understood as limits.

The differential operator under consideration in this paper is the Laplacian in $L^2(G)$ equipped with continuity and Kirchhoff conditions at all vertices, i.e. the *Kirchhoff Laplacian* $-\Delta_G$ in $L^2(G)$ given by

$$(-\Delta_G f)_e = -f''_e, \quad e \in E,$$

$$\text{dom}(-\Delta_G) = \left\{ f \in H^1(G) \cap \tilde{H}^2(G) : \partial_\nu f(v) = 0, v \in V \right\}.$$

The operator $-\Delta_G$ is selfadjoint in $L^2(G)$ and its spectrum $\sigma(-\Delta_G)$ is nonnegative and consists of isolated eigenvalues with finite multiplicities; the smallest eigenvalue of $-\Delta_G$ equals zero and corresponds to those eigenfunctions which are constant on G .

In the following the notion of a (real) resonance of G plays an important role. We make the following definition.

Definition 2.1. Let $-\Delta_G$ be the Kirchhoff Laplacian in $L^2(G)$. A number $\lambda \in \mathbb{R}$ is called *(real) resonance of G* if there exists a nontrivial function $f \in \ker(-\Delta_G - \lambda)$ such that $f(v) = 0$ holds for each vertex v of G . Each such f is called *resonance eigenfunction*.

Some remarks are in order. The classical definition of a resonance for (the Laplacian on) a metric graph refers to graphs which contain infinite leads, i.e. edges of infinite length attached to some vertices; see, e.g., [9]. A resonance for such a non-compact graph is defined as a number $k \in \mathbb{C} \setminus \{0\}$ such that there exists a nontrivial solution f of $-f'' = k^2 f$ on G which satisfies the continuity and Kirchhoff condition at each vertex and is a scalar multiple of e^{ikx} on each infinite lead. It can be seen easily that a resonance must satisfy $\operatorname{Im} k \leq 0$, and it may happen that a resonance k is real. If G is a finite metric graph and \tilde{G} is the non-compact graph obtained from attaching infinite leads to each vertex of G then it can be shown that k is a real resonance of \tilde{G} in the classical sense if and only if $\lambda = k^2$ is a resonance of G in the sense of Definition 2.1. We work with $\lambda = k^2$ instead of k since for our purposes this choice is more convenient. Note that also with respect to our definition a resonance of G is nonzero (and, thus, positive) as the eigenfunctions of $-\Delta_G$ corresponding to $\lambda = 0$ are necessarily different from zero at each vertex.

3. RESONANCES ON THE REAL LINE

In this section we provide an explicit formula for the smallest real resonance of a finite metric graph G in terms of its connectivity and algebraic properties of the edge lengths.

In order to formulate the main result of this section some preparation is needed. For any closed walk

$$W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{w-1}, e_w, v_w) \quad (3.1)$$

in G we denote by

$$L_W := L(e_1) + \dots + L(e_w)$$

its total length. Note that for each closed walk of the form (3.1) exactly one of the following alternatives holds true: either two of the edges e_1, \dots, e_w have rationally independent lengths or the edges e_1, \dots, e_w are commensurate, i.e. pairwise rationally dependent. In the latter case there exists (a non-unique) $u \in \mathbb{R}$ with $u > 0$ such that $n_j := n_j(u) := L(e_j)/u$ are natural numbers for $j = 1, \dots, w$. We denote the largest such u by u_W . Note that it can be expressed as

$$u_W = u \cdot \gcd(n_1, \dots, n_w) \quad (3.2)$$

for any u with the above property. In other words, u_W is the largest unit of length with respect to which the lengths of all edges in W are integers. Note that $W_1 \subset W_2$ implies $u_{W_2} \leq u_{W_1}$. According to the definition of u_W we say that

$$W \text{ has even (odd) length if } \frac{L_W}{u_W} \text{ is even (odd).}$$

In the statement of the following theorem we say that a closed walk W in G is *non-multiple* if there exists an edge which appears exactly once in W . Note that, in particular, each cycle is a non-multiple closed walk. Moreover, we say that a cycle C has *commensurate edges* if all its edge lengths are commensurate. For an illustration of the conditions in the definition of $\nu(C)$ we refer the reader to Example 3.4 below.

Theorem 3.1. *Let G be a finite, connected metric graph. For each cycle C in G with commensurate edges define*

$$\nu(C) = \begin{cases} 1, & \text{if there exists a non-multiple closed walk } W \supset C \text{ with} \\ & \text{commensurate edges and even length such that } u_W = u_C, \\ 2, & \text{otherwise.} \end{cases}$$

If the expression

$$\lambda_G := \inf \left\{ \frac{\nu(C)^2 \pi^2}{u_C^2} : C \text{ cycle in } G \text{ with commensurate edges} \right\}$$

is finite then λ_G is the smallest real resonance of G ; otherwise G has no real resonance.

Proof. Note first that $\lambda_G = \infty$ if and only if G does not contain any cycle with commensurate edges. In this case it follows from a reasoning as in [22, Section 4] that no resonances exist. Therefore from now on we assume $\lambda_G < \infty$. The following main part of the proof is carried out in two steps. In Step 1 we show that any real resonance of G satisfies $\lambda \geq \lambda_G$. Afterwards in Step 2 we verify that the number λ_G defined in the theorem is always a resonance of G .

Step 1. In this step we show that each resonance of G is located above λ_G . For this let $\lambda \in \mathbb{R}$ be a resonance of G and let $f \in \ker(-\Delta_G - \lambda)$ be a corresponding resonance eigenfunction, i.e., $f(v) = 0$ for all $v \in V$; without loss of generality we assume that f is real-valued. We distinguish two cases; for this note that the support of f contains at least one cycle and that each cycle contained in the support of f has commensurate edges; cf. [22, Section 4].

Case 1: $\nu(C) = 1$ for at least one cycle C contained in the support of f . Let that cycle be given by $C = (v_0, e_1, v_1, e_2, v_2, \dots, v_{c-1}, e_c, v_c)$. Then there exist nonzero $\alpha_1, \dots, \alpha_c \in \mathbb{R}$ such that

$$f_{e_j}(x) = \alpha_j \sin(\sqrt{\lambda}x), \quad x \in e_j,$$

holds for $j = 1, \dots, c$. In particular, due to $f_{e_j}(L(e_j)) = 0$ for $j = 1, \dots, c$, there exist $n_1, \dots, n_c \in \mathbb{N}$ such that

$$\sqrt{\lambda}L(e_j) = n_j\pi, \quad j = 1, \dots, c,$$

or, equivalently, $L(e_j) = un_j$, $j = 1, \dots, c$, with $u := \pi/\sqrt{\lambda}$. Thus with the help of (3.2) we obtain

$$\lambda = \frac{\pi^2}{u^2} \geq \frac{\pi^2}{u^2 \gcd(n_1, \dots, n_c)^2} = \frac{\nu(C)^2 \pi^2}{u_C^2} \geq \lambda_G$$

as $\nu(C) = 1$. This is the desired assertion in this case.

Case 2: $\nu(C) = 2$ for each cycle C contained in the support of f . We construct an explicit cycle and a corresponding non-multiple closed walk related to the properties of f . In the following for any $e \in E$ we write

$$\partial_\nu f_e(v) := \begin{cases} -f'_e(0), & \text{if } v = o(e) \neq t(e), \\ f'_e(L(e)), & \text{if } v = t(e) \neq o(e), \\ f'_e(L(e)) - f'_e(0), & \text{if } v = o(e) = t(e), \end{cases}$$

for the derivative of f_e at v pointing outwards. Note that with this abbreviation the Kirchhoff condition $\partial_\nu f(v) = 0$ at a vertex v can be rewritten as

$$\sum_{e \text{ incident to } v} \partial_\nu f_e(v) = 0. \quad (3.3)$$

Our construction of a non-multiple closed walk is done by carrying out the following algorithm.

- (o) Choose an arbitrary edge e_1 in the support of f and a vertex v_0 to which e_1 is incident. Set $w = 1$.
 (a) Define the vertex v_w by

$$v_w = \begin{cases} t(e_w), & \text{if } o(e_w) = v_{w-1}, \\ o(e_w), & \text{else.} \end{cases}$$

- (b) If there exists $p \leq w$, such that $v_w = v_{p-1}$ and

$$\operatorname{sgn}(\partial_\nu f_{e_w}(v_w)) = -\operatorname{sgn}(\partial_\nu f_{e_p}(v_{p-1}))$$

then go to (d); otherwise go to (c).

- (c) Choose an edge e_{w+1} incident to v_w such that

$$\operatorname{sgn}(\partial_\nu f_{e_w}(v_w)) = -\operatorname{sgn}(\partial_\nu f_{e_{w+1}}(v_w)).$$

Increase w by 1 and return to (a).

- (d) Define the closed walk W by

$$W := (v_{p-1}, e_p, v_p, \dots, v_{w-1}, e_w, v_w).$$

Observe that the choice of an edge e_{w+1} with the properties required in step (c) is always possible and that each of the edges e_j appearing in the algorithm belongs to the support of f . Indeed, the starting edge e_1 belongs to the support of f , that is, $f_{e_1}(x) = \alpha_1 \sin(\sqrt{\lambda}x)$, $x \in e_1$, for some nonzero $\alpha_1 \in \mathbb{R}$; in particular, the derivative of f_{e_1} at each endpoint of e_1 is nonzero, and if $\partial_\nu f_{e_1}(v_1) = 0$ then $v_1 = v_0$ and the condition of (b) is satisfied, that is, the algorithm terminates. If $\partial_\nu f_{e_1}(v_1) \neq 0$ then by the Kirchhoff condition (3.3) there exists an edge e_2 incident to v_1 which satisfies the condition in (c). In particular, $\partial_\nu f_{e_2}(v_1) \neq 0$, that is, $f_{e_2} \neq 0$. In the same way it follows successively that the choice of e_{w+1} in step (c) is always possible, that $\partial_\nu f_{e_{w+1}}(v_w) \neq 0$, and that e_{w+1} belongs to the support of f for all w . In particular, there exist nonzero $\alpha_p, \dots, \alpha_w \in \mathbb{R}$ such that

$$f_{e_j}(x) = \alpha_j \sin(\sqrt{\lambda}x), \quad x \in e_j, \quad j = p, \dots, w. \quad (3.4)$$

Moreover, the construction in step (c) and the condition in (b) imply

$$\operatorname{sgn}(\partial_\nu f_{e_j}(v_j)) = -\operatorname{sgn}(\partial_\nu f_{e_{j+1}}(v_j)), \quad j = p, \dots, w, \quad (3.5)$$

where we have identified e_p with an “imaginary” edge e_{w+1} . In particular, it follows with the help of (3.5) that

$$e_j \neq e_k \quad \text{if } k = j + 1. \quad (3.6)$$

Note further that the above algorithm terminates at the latest when visiting a vertex for the third time as in this case the condition of (b) is satisfied necessarily.

It is our next aim to show that the walk W is non-multiple and contains a cycle C . As a preparation for this note that

$$e_l = e_k \text{ for } l \neq k \text{ implies } v_{l-1} = v_k \text{ and } v_l = v_{k-1}. \quad (3.7)$$

Indeed, suppose $e_l = e_k$ for $l < k$ with $v_{l-1} = v_{k-1}$. Then by (3.5) we have

$$\operatorname{sgn}(\partial_\nu f_{e_{k-1}}(v_{k-1})) = -\operatorname{sgn}(\partial_\nu f_{e_k}(v_{k-1})) = -\operatorname{sgn}(\partial_\nu f_{e_l}(v_{l-1})).$$

Thus the condition of (b) with $w = k - 1$ and $p = l$ is satisfied and the algorithm would have stopped with the vertex v_{k-1} which contradicts the existence of e_k in W . Note that, as a consequence, each edge appears at most twice in W . Let now c be the index of the first repeated vertex in W , i.e. the smallest index in $\{p, \dots, w\}$ such that there exists an index $q \in \{p, \dots, c\}$ with $v_{q-1} = v_c$. Then clearly

$$C := (v_{q-1}, e_q, v_q, \dots, v_{c-1}, e_c, v_c) \subset W \quad (3.8)$$

is a closed walk whose vertices v_q, \dots, v_c are distinct, i.e., C is a cycle. It remains to show that W is non-multiple. For this consider two cases: First, let $e_q = e_c$. If $q < c$ then by (3.7)

$$v_{(q+1)-1} = v_q = v_{c-1}. \quad (3.9)$$

On the one hand, $q+1 \leq c-1$ and (3.9) would contradict the choice of the index c . On the other hand, $q+1 = c$ would imply $e_{c-1} = e_q = e_c$, which contradicts (3.6). Thus, as $q \leq c$ it follows $q = c$, i.e., C is a loop. Assume that there exists $j \neq c$ such that $e_j = e_c$. Observe that $j < c$ is impossible; it would imply $j < q$ and $v_{j-1} = v_c = v_{q-1}$ (see (3.7)), and the construction of C would stop with v_{q-1} at the latest, a contradiction to the fact that C starts with the index $q-1$. Thus $j > c$. Moreover, $j = c+1$ would imply $e_c = e_j = e_{c+1}$, which again contradicts (3.6). Hence $c-1 < c < j-1 < j$. Since $v_{c-1} = v_c = v_{j-1} = v_j$, this contradicts the construction of W . Second, consider the case $e_q \neq e_c$, i.e., $q < c$. Assume for a contradiction that $e_c = e_j$ for some $j \neq c$. Then

$$v_{j-1} = v_c = v_{q-1} \quad (3.10)$$

by (3.7) and the definition of C . Observe that all indices in (3.10) are distinct. In fact, $j-1 = c$ would imply $e_{j-1} = e_c = e_j$, which is a contradiction to (3.6), $j-1 = q-1$ would imply $e_c = e_j = e_q$, and $q-1 < c-1 < c$. In fact, we even have $j-1 > c$ since $j-1 < c$ together with (3.10) would contradict the choice of c . Hence due to (3.10) the construction algorithm of W would terminate with v_{j-1} and thus e_{j-1} would be the last edge in W , a contradiction. Consequently, W is a non-multiple closed walk with commensurate edges and contains the cycle C in (3.8).

Let us now prove the claim $\lambda \geq \lambda_G$. Note that by (3.4) there exist $n_p, \dots, n_w \in \mathbb{N}$ such that

$$\sqrt{\lambda}L(e_j) = n_j\pi, \quad j = p, \dots, w;$$

in particular, $L(e_j) = n_j u$ for $j = p, \dots, w$ with $u := \pi/\sqrt{\lambda}$. Since $\nu(C) = 2$ by assumption and $W \supset C$ is a non-multiple closed walk with commensurate edges, it follows that $u_W < u_C$ or W has odd length. If the first holds then by (3.2)

$$\gcd(n_p, \dots, n_w) = \frac{u_W}{u} < \frac{u_C}{u} = \gcd(n_q, \dots, n_c),$$

and as $\gcd(n_p, \dots, n_w)$ is a divisor of $\gcd(n_q, \dots, n_c)$ we can conclude

$$2 \gcd(n_p, \dots, n_w) \leq \gcd(n_q, \dots, n_c).$$

Hence

$$\lambda = \frac{\pi^2}{u^2} \geq \left(\frac{\pi}{u \gcd(n_p, \dots, n_w)} \right)^2 \geq \left(\frac{2\pi}{u \gcd(n_q, \dots, n_c)} \right)^2 = \frac{\nu(C)^2 \pi^2}{u_C^2} \geq \lambda_G$$

by the definition of λ_G .

Let us now come to the case where W has odd length. Consider the graph G_W consisting of one simple cycle of length L_W obtained by arranging the edges of the walk W to a cycle while keeping the order of the edges; more specifically, G_W is the graph consisting of the distinct edges $\tilde{e}_p, \dots, \tilde{e}_w$ where \tilde{e}_j is a copy of e_j , $j = p, \dots, w$, and the distinct vertices $\tilde{v}_p, \dots, \tilde{v}_w$ of degree two such that the edges incident to \tilde{v}_j are \tilde{e}_j and \tilde{e}_{j+1} for $j = p, \dots, w-1$ or \tilde{e}_w and \tilde{e}_p for $j = w$. We assume that the orientation of the edges of G_W is chosen such that $t(\tilde{e}_j) = \tilde{v}_j$ in G_W if and only if $t(e_j) = v_j$ in G . (Note that if an edge in W is a loop then the corresponding edge in G_W is not a loop and its orientation in G_W can be chosen arbitrarily.) For $j = p, \dots, w$, define

$$\tilde{f}_{e_j}(x) = \operatorname{sgn}(\alpha_j) \sin(\sqrt{\lambda}x), \quad x \in \tilde{e}_j,$$

see (3.4), that is, \tilde{f} can be viewed as a scaled version of the restriction of the original eigenfunction f to W . Then \tilde{f} vanishes at each vertex and it follows immediately from (3.5) that \tilde{f} satisfies the Kirchhoff condition at each vertex of G_W . Thus \tilde{f} is an eigenfunction of the Kirchhoff Laplacian $-\Delta_{G_W}$ on the graph G_W . Since G_W can be identified with a loop of length L_W it follows

$$\sqrt{\lambda}L_W \in 2\pi\mathbb{N}. \quad (3.11)$$

On the other hand, as W has odd length, L_W/u_W is odd and thus

$$\sqrt{\lambda}L_W = \pi \frac{L_W}{u} \geq \pi \frac{L_W}{u_W} \notin 2\pi\mathbb{N}.$$

From this together with (3.11) it follows

$$u < u_W = u \gcd(n_p, \dots, n_w).$$

In particular, $\gcd(n_q, \dots, n_c) \geq \gcd(n_p, \dots, n_w) \geq 2 = \nu(C)$ and hence

$$\lambda = \frac{\pi^2}{u^2} \geq \frac{\nu(C)^2 \pi^2}{u^2 \gcd(n_q, \dots, n_c)^2} = \frac{\nu(C)^2 \pi^2}{u_C^2} \geq \lambda_G,$$

where again the definition of λ_G was employed. Thus in any case we have $\lambda \geq \lambda_G$.

Step 2. In this second step we show that λ_G is in fact a resonance. This is done via an explicit construction of a resonance eigenfunction. Let

$$C = (v_{q-1}, e_q, v_q, \dots, v_{c-1}, e_c, v_c)$$

be a cycle in G with commensurate edges such that

$$\lambda_G = \frac{\nu(C)^2 \pi^2}{u_C^2}. \quad (3.12)$$

Furthermore, let us choose a non-multiple closed walk with commensurate edges

$$W = (v_{p-1}, e_p, v_p, e_{p+1}, v_{p+1}, \dots, v_{w-1}, e_w, v_w) \supset C$$

with $u_W = u_C$ and such that W has even length if $\nu(C) = 1$; in fact, for $\nu(C) = 1$ this is possible due to the definition of $\nu(C)$ and in the case $\nu(C) = 2$ the choice $C = W$ has the desired properties. Then (3.12) and $u_W = u_C$ imply

$$\sqrt{\lambda_G} \sum_{j=p}^k L(e_j) = \nu(C) \pi \frac{\sum_{j=p}^k L(e_j)}{u_W} \in \pi\mathbb{N}, \quad k = p, \dots, w, \quad (3.13)$$

since $L(e_j)/u_W$ is a natural number for $j = p, \dots, w$. Moreover,

$$\sqrt{\lambda_G} L_W = \nu(C) \pi \frac{L_W}{u_W} \in 2\pi\mathbb{N} \quad (3.14)$$

as L_W/u_W is even if $\nu(C) = 1$ and $\nu(C) = 2$ otherwise. Let now G_W be the graph consisting of one simple cycle obtained by arranging the edges of the walk W to a cycle according to their order in W ; cf. Step 1. Identify the function $\sin(\sqrt{\lambda_G}x)$, $x \in [0, L_W]$, in the obvious way with a function $\tilde{f} : G_W \rightarrow \mathbb{R}$, where we identify the endpoints of $[0, L_W]$ with the vertex v_w and traverse G_W according to increasing edge indices, i.e., in the order of the original walk W . Then (3.13) implies $\tilde{f}(\tilde{v}) = 0$ for each vertex \tilde{v} of G_W . Moreover, by construction \tilde{f} satisfies the Kirchhoff condition at each vertex different from \tilde{v}_w , and the validity of the Kirchhoff condition at \tilde{v}_w is guaranteed by (3.14). Thus \tilde{f} is a resonance eigenfunction of the cycle graph G_W . “Spooling” this function onto the graph G , i.e. identifying \tilde{f} with the corresponding function f on W (where sums must be taken on edges appearing twice in W) and setting $f = 0$ identically outside of W , leads to a function $f \in \ker(-\Delta_G - \lambda_G)$ satisfying $f(v) = 0$ for all $v \in V$. Moreover, f is nontrivial as

W is non-multiple. Thus f is a resonance eigenfunction for G and, hence, λ_G is a resonance. This completes the proof of the theorem. \square

An immediate corollary of Theorem 3.1 looks as follows. Here the calculation of the numbers $\nu(C)$ is avoided.

Corollary 3.2. *Let G be a finite, connected metric graph and let*

$$\tilde{\lambda}_G := \inf \left\{ \frac{\pi^2}{u_C^2} : C \text{ cycle in } G \text{ with commensurate edges} \right\}.$$

Then G has no resonance in $(-\infty, \tilde{\lambda}_G)$.

We illustrate the application of Theorem 3.1 by some examples.

Example 3.3. Let G be the metric graph given on the left-hand side of Figure 1 and assume that all edges have equal length π . Our aim is to calculate the smallest real resonance of G with the help of Theorem 3.1. First of all, up to shifts and

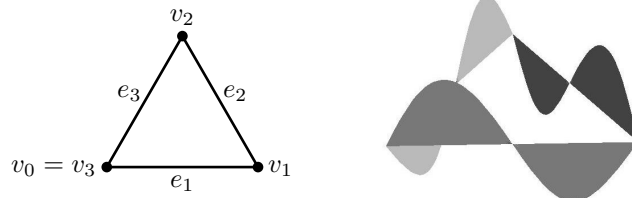


FIGURE 1. An equilateral metric graph and a resonance eigenfunction corresponding to the smallest real resonance.

inversion the only cycle in G is given by $C = (v_0, e_1, v_1, e_2, v_2, e_3, v_3)$. Setting $u = \pi$ and $n_1 = n_2 = n_3 = 1$ we have $L(e_j) = n_j u$ for $j = 1, 2, 3$. Thus

$$u_C = \pi \cdot \gcd(n_1, n_2, n_3) = \pi.$$

Up to shift and inversion, C is also the only non-multiple closed walk which contains C , and C has odd length since $L_C/u_C = 3$. Hence $\nu(C) = 2$ and

$$\lambda_G = \left(\frac{\nu(C)\pi}{u_C} \right)^2 = 4$$

is the smallest real resonance of G . When parametrizing the graph such that $t(e_j) = v_j$, $j = 1, 2, 3$, the (up to multiples unique) resonance eigenfunction of $-\Delta_G$ corresponding to $\lambda_G = 4$ is given by

$$f_{e_j}(x) = \sin(2x), \quad x \in e_j, \quad j = 1, 2, 3,$$

and is displayed on the right-hand side of Figure 1. A simple calculation yields that $0, 4/9$ and $16/9$ are the eigenvalues of $-\Delta_G$ below λ_G , which are no resonances.

A slightly more involved situation is considered in the following example.

Example 3.4. Let G be the metric graph given on the left-hand side of Figure 2 and assume that all edges have equal length π . Our aim is again to calculate the smallest real resonance by means of Theorem 3.1. Up to shifts and inversion the only two cycles in G are given by $C_1 = (v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ and $C_2 = (v_4, e_5, v_5, e_6, v_6, e_7, v_7)$. Setting $u = \pi$ and $n_1 = \dots = n_7 = 1$ we have $L(e_j) = n_j u$, $j = 1, \dots, 7$, and thus

$$u_{C_1} = \pi \cdot \gcd(n_1, n_2, n_3) = \pi.$$

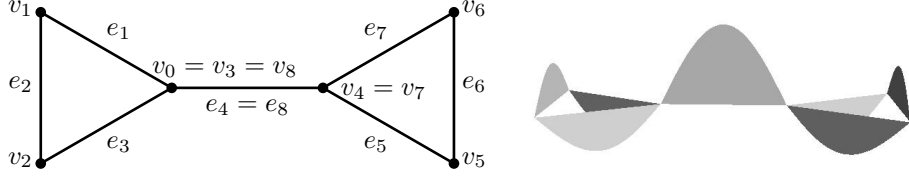


FIGURE 2. An equilateral metric graph with two cycles and a resonance eigenfunction corresponding to the smallest real resonance.

Analogously, $u_{C_2} = \pi$. Both cycles C_1 and C_2 are contained in the non-multiple closed walk

$$W = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4, e_5, v_5, e_6, v_6, e_7, v_7, e_8, v_8),$$

and as above we find $u_W = \pi = u_{C_1} = u_{C_2}$. Moreover, $L_W/u_W = 8$, i.e., W has even length. Therefore $\nu(C_1) = \nu(C_2) = 1$ and

$$\frac{\nu(C_1)^2 \pi^2}{u_{C_1}^2} = \frac{\nu(C_2)^2 \pi^2}{u_{C_2}^2} = \frac{\pi^2}{\pi^2} = 1,$$

which implies $\lambda_G = 1$. Parametrizing G in such a way that $t(e_j) = v_j$, $j = 1, \dots, 7$, we can write a corresponding resonance eigenfunction f as

$$f_{e_j}(x) = \sin(x), \quad x \in e_j, \quad j = 2, 6,$$

$$f_{e_j}(x) = -\sin(x), \quad x \in e_j, \quad j = 1, 3, 5, 7,$$

and

$$f_{e_7}(x) = 2 \sin(x), \quad x \in e_4 = e_8;$$

see the right-hand side of Figure 2.

Remark 3.5. In the previous example each cycle has odd length, but there exists a non-multiple closed walk which contains both cycles and has even length. This shows that the definition of $\nu(C)$ in Theorem 3.1 cannot be simplified: it does not suffice to consider only the length of each cycle C itself.

The following last example of this section demonstrates that small perturbations of the edge lengths may have a large influence on the smallest real resonance. Moreover, it can be seen from the following example that a considerable number of eigenvalues may lie below the smallest resonance of a metric graph.

Example 3.6. Let G be the metric graph given in Figure 3. As a simple calculation

$L(e_2)$	λ_G	#EV $< \lambda_G$
3.14159	$(\frac{\pi}{3.14159})^2 \approx 1$	1
3.14161	$\pi^2 \cdot 10^{10}$	628319
3.14160	$4\pi^2 \cdot 10^{10}$	1256637
π	∞	∞

FIGURE 3. A metric graph G and a table with λ_G and the number of eigenvalues (counted with multiplicities) below λ_G depending on the choice of $L(e_2)$. The length of e_1 is fixed by $L(e_1) = 3.14159$.

shows, the eigenvalues of $-\Delta_G$ are given by the numbers

$$\frac{4k^2\pi^2}{(L(e_1) + L(e_2))^2}, \quad k \in \mathbb{N} \cup \{0\}, \quad (3.15)$$

where the eigenvalue zero has multiplicity one and all further eigenvalues have multiplicity two. In this example we fix the length of e_1 to be $L(e_1) = 3.14159$ and vary the length of e_2 . A calculation similar to those in Example 3.3 and Example 3.4 then yields the smallest real resonance λ_G , and the number of eigenvalues of $-\Delta_G$ below λ_G can then be calculated from (3.15). The results for several choices of $L(e_2)$ are displayed in the table in Figure 3; they show that small perturbations of the edge length may strongly influence the smallest real resonance as well as the number of non-resonant eigenvalues below λ_G .

4. VISIBILITY OF QUANTUM GRAPH SPECTRUM FROM THE VERTICES

In this section we apply the result of Section 3 to the question of visibility of quantum graph spectrum from vertex data which is given via the Titchmarsh–Weyl function.

We start by recalling the definition of the Titchmarsh–Weyl function. For its well-definedness see, e.g., [22, Lemma 2.2].

Definition 4.1. Let $B = \{v_1, \dots, v_m\}$ be a nonempty subset of the set of vertices of G . Moreover, let $\mu \in \mathbb{C} \setminus \sigma(-\Delta_G)$. For $l = 1, \dots, m$ let $f^{(l)} \in \tilde{H}^2(G) \cap H^1(G)$ with $-f_e^{(l)''} = \mu f_e^{(l)}$, $e \in E$, such that $\partial_\nu f^{(l)}(v_l) = 1$ and

$$\partial_\nu f^{(l)}(v) = 0, \quad v \in V \setminus \{v_l\}.$$

The *Titchmarsh–Weyl matrix* $M_B(\mu) \in \mathbb{C}^{m \times m}$ is the matrix with the entries

$$(M_B(\mu))_{k,l} = f^{(l)}(v_k), \quad k, l = 1, \dots, m.$$

The matrix function $\mu \mapsto M_B(\mu)$ is called *Titchmarsh–Weyl function*.

Obviously the Titchmarsh–Weyl function depends on the chosen vertex set B . Below we will fix B sufficiently large for our purposes. Note that the Titchmarsh–Weyl function satisfies

$$M_B(\mu) \begin{pmatrix} \partial_\nu f(v_1) \\ \vdots \\ \partial_\nu f(v_m) \end{pmatrix} = \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_m) \end{pmatrix}, \quad \mu \in \mathbb{C} \setminus \sigma(-\Delta_G),$$

for any solution $f \in \tilde{H}^2(G) \cap H^1(G)$ of $-f_e'' = \mu f_e$, $e \in E$, such that $\partial_\nu f(v) = 0$ for all $v \in V \setminus B$. For fixed $\mu \in \mathbb{C} \setminus \sigma(-\Delta_G)$ the matrix $M_B(\mu) \in \mathbb{C}^{m \times m}$ can be regarded as a Neumann-to-Dirichlet map for the graph Laplacian.

In the following for an analytic matrix function R defined on an open set $\Omega \subset \mathbb{C}$ we say that $\lambda \in \mathbb{C}$ is a pole of order n of R if there exists an open neighborhood \mathcal{O} of λ in \mathbb{C} such that $\mathcal{O} \setminus \{\lambda\} \subset \Omega$,

$$\lim_{\mu \rightarrow \lambda} (\mu - \lambda)^n R(\mu) \text{ exists and is nontrivial, and } \lim_{\mu \rightarrow \lambda} (\mu - \lambda)^{n+1} R(\mu) = 0.$$

We say that R is meromorphic if $\mathbb{C} \setminus \Omega$ consists of isolated points which are poles or removable singularities. For any $\lambda \in \mathbb{C}$ we denote by $\text{Res}_\lambda R$ the residue of R at λ , i.e., the matrix given by

$$\text{Res}_\lambda R = \frac{1}{2\pi i} \int_\Gamma R(\mu) d\mu,$$

where Γ is any closed Jordan curve in Ω which surrounds λ but no other point in $\mathbb{C} \setminus \Omega$. In particular, if $\lambda \in \Omega$ or if λ is a removable singularity of R then $\text{Res}_\lambda R = 0$. Moreover, $\lambda \in \mathbb{C} \setminus \Omega$ is a pole of R if and only if $\text{rank Res}_\lambda R > 0$.

For the following proposition see [22, Proposition 2.4].

Proposition 4.2. *Let $B = \{v_1, \dots, v_m\}$ be any nonempty set of vertices of G . Then the matrix-valued Titchmarsh–Weyl function $\mu \mapsto M_B(\mu)$ in Definition 4.1 is meromorphic on \mathbb{C} with possible poles of order one in the discrete set $\sigma(-\Delta_G)$. Moreover, for each $\lambda \in \mathbb{R}$ the identity*

$$\dim \ker(-\Delta_G - \lambda) = \text{rank Res}_\lambda M_B + \dim K_\lambda$$

holds, where

$$K_\lambda := \{f \in \ker(-\Delta_G - \lambda) : f(v) = 0 \text{ for all } v \in B\}.$$

It follows immediately from Proposition 4.2 that each pole of M_B is an eigenvalue of $-\Delta_G$. However, eigenvalues may occur which do not materialize as poles of M_B .

For a suitable choice of the vertex set B in order to apply Theorem 3.1 let us recall the following definition; cf. Figure 4.

Definition 4.3. Let G be a finite metric graph.

- (i) A *boundary vertex* is a vertex of degree one.
- (ii) The *core* of G is the largest subgraph of G which does not have boundary vertices.
- (iii) We call a vertex v of G *proper core vertex* if all edges attached to v belong to the core of G .

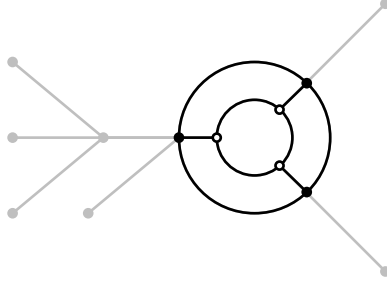


FIGURE 4. A graph with its core marked in black and the proper core vertices drawn with empty dots.

We are going to consider the Titchmarsh–Weyl function for any vertex set B which contains all boundary vertices and all proper core vertices of G . As a preparation we show the following simple lemma.

Lemma 4.4. *Let B be a set of vertices which contains all boundary vertices and all proper core vertices of G . Let $\lambda \in \mathbb{R}$ and let $f \in \ker(-\Delta_G - \lambda)$ such that $f(v) = 0$ for all $v \in B$. Then $f(v) = 0$ for all $v \in V$.*

Proof. The graph G consists of its core and a finite number of rooted trees (i.e. graphs without cycles) with their roots belonging to the core of G , and if T is any of these trees, by assumption, $f(v) = 0$ for all but at most one boundary vertex v of T . Thus a reasoning as in Step 1 of the proof of [22, Theorem 3.5] yields $f = 0$ identically on T , in particular, $f(v) = 0$ for each vertex v of T . Since $f(v) = 0$ for each proper core vertex v of G , it follows $f(v) = 0$ for all $v \in V$. \square

We are now able to formulate the main result of this section.

Theorem 4.5. *Let G be a finite, connected metric graph and let $B \subset V$ be a set of vertices which contains all boundary vertices and all proper core vertices of G . Define λ_G as in Theorem 3.1. Then the following assertions hold.*

- (i) Let $\lambda < \lambda_G$. Then λ is an eigenvalue of $-\Delta_G$ if and only if λ is a pole of M_B . Moreover,

$$\dim \ker(-\Delta_G - \lambda) = \text{rank Res}_\lambda M_B$$

holds.

- (ii) The number $\lambda = \lambda_G$ is an eigenvalue of $-\Delta_G$ and the strict inequality

$$\dim \ker(-\Delta_G - \lambda) > \text{rank Res}_\lambda M_B$$

is satisfied.

Proof. Let $\lambda < \lambda_G$. By Theorem 3.1 the space K_λ in Proposition 4.2 is trivial. Indeed, by Lemma 4.4 each $f \in K_\lambda$ satisfies $f(v) = 0$ for all $v \in V$ and since λ is not a resonance it follows $f = 0$ identically on G . Thus Proposition 4.2 yields the assertions of item (i). For $\lambda = \lambda_G$ by Theorem 3.1 there exists a resonance eigenfunction f ; in particular, $f \in K_\lambda$, that is, $\dim K_\lambda \geq 1$. Thus Proposition 4.2 implies the assertion (ii) of the theorem. \square

In the spirit of Corollary 3.2 we obtain the following consequence.

Corollary 4.6. *Let G be a finite, connected metric graph and define $\tilde{\lambda}_G$ as in Corollary 3.2. Furthermore, let $\lambda < \tilde{\lambda}_G$. Then λ is an eigenvalue of $-\Delta_G$ if and only if λ is a pole of M_B . Moreover,*

$$\dim \ker(-\Delta_G - \lambda) = \text{rank Res}_\lambda M_B$$

holds.

We close this paper with an example which shows that in some cases the eigenvalue λ_G of $-\Delta_G$ is in fact a removable singularity of M_B , that is, λ_G is invisible for M_B .

Example 4.7. Consider the metric graph in Figure 5 which consists of a loop and a further edge attached to it. We assume $L(e_1) = \pi$ and $L(e_2) = 1$. Then an

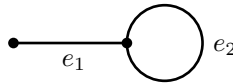


FIGURE 5. The metric graph G in Example 4.7.

application of Theorem 3.1 yields that the smallest real resonance of G is given by

$$\lambda_G = 4\pi^2.$$

Furthermore, a simple calculation yields that the corresponding eigenspace of $-\Delta_G$ is one-dimensional and is spanned by the function f with

$$f_{e_1}(x) = 0, \quad x \in e_1, \quad \text{and} \quad f_{e_2}(x) = \sin(2\pi x), \quad x \in e_2.$$

In particular, for any choice of B we have

$$K_\lambda = \text{span}\{f\} = \ker(-\Delta_G - \lambda_G)$$

and Proposition 4.2 yields $\text{rank Res}_\lambda M_B = 0$. Hence the eigenvalue λ_G is a removable singularity of M_B or, in other words, it is invisible for M_B .

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